

Newton Method online class

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Let f be a continuous differentiable function. We start with a first approximation x_1 of a root of f , which is obtained by guessing or from a rough sketch of the graph of f , or from a computer generated graph of f . Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the intercept of L , labeled x_2 . The idea behind Newton's method is that the tangent line L is close to the curve and so its x -intercept, x_2 , is close to the x -intercept of the curve (namely, the root r that we are seeking). Because the tangent is a line, we can easily find its x intercept.

To find a formula for x_2 in terms of x_1 we use the fact that the slope of L is $f'(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Since the x -intercept of L is x_2 , we set $y = 0$ and obtain

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2 :

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second approximation to r .

Next we repeat this procedure with x_1 replaced by the second approximation x_2 , using the tangent line at $(x_2, f(x_2))$. This gives the third approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

If we keep repeating this process, we obtain a sequence of approximation $x_1, x_2, x_3, x_4, \dots$. In general, if the n th approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (*)$$

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence converges to r and we write

$$\lim_{n \rightarrow \infty} x_n = r$$

To illustrate the newton method you can see the following figure

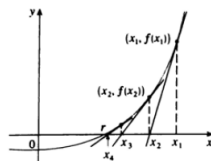


FIGURE 3

Important note : Sometimes the Newton's method fails or works very slowly and a better initial approximation x_1 should be chosen. When $f'(x_1)$ is close to 0 for instance then x_2 could be a worse approximation than x_1 as in the bellow figure or even fall outside the domain of f .

You can perhaps understand how this can happen by observing the following picture :

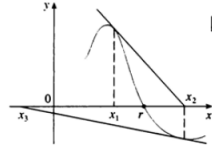


FIGURE 4

Let see some example for the Newton Method and make sure you know what to do and how to use it.

Exercise : Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

Solution : In order to apply Newton method and we need to use formula (*) and thus need to know what is playing the role of f in this exercise. Here the problem is about finding the roots of $x^3 - 2x - 5$ thus we should certainly consider $f(x) = x^3 - 2x - 5$. Now in order to apply formula (*) we need the derivative of f . Let's compute it.

$$f'(x) = 3x^2 - 2$$

Newton himself used this equation to illustrate his method and he chose $x_1 = 2$ because he knew that $f(1) = -6$, $f(2) = -1$ and $f(3) = 16$.

Formula (*) becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

With $n = 1$, we have

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2} = 2 - \frac{2^3 - 2 \times 2 - 5}{3 \times 2^2 - 2} \approx 2.1$$

Then with $n = 2$ we obtain

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2} = 2.1 - \frac{2.1^3 - 2 \times 2.1 - 5}{3 \times 2.1^2 - 2} \approx 2.0946$$

It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places.

Suppose that we want to achieve a given accuracy, say to eight decimal places, using Newton's method. How do we know when to stop? The rule of thumb that is generally used is that we can stop when successive approximations x_n and x_{n+1} agree to eight decimal places.

Exercise : Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

Solution : We need to translate this problem into a problem where we can apply Newton Method. For this we need a function f to consider and translate the problem into finding the roots of this function. Now, remember that

$$\sqrt[6]{2}^6 = 2$$

by definition. What does this means, it means that $\sqrt[6]{2}$ is a root for the polynomial $x^6 - 2$. Thus we could consider f to be $f(x) = x^6 - 2$. We also need a good approximation of the root. We know that $f(1) = -1$ and $f(2) = 62$, thus using the intermediate value Theorem we know that the root is between 1 and 2 but since $f(2)$ is so far from zero compared to $f(1)$, so 1 is a much better approximation than

$f(2)$. So we should probably take $x_1 = 1$. Now we need to compute $f'(x)$ because it is required in the formula (*). But that is super easy, $f'(x) = 6x^5$. And now we are going to apply Newton's method successively until we get two number with same 8 digits after the comma.

If you do so you get from $x_1 = 1$ the first approximation that we choosed, you should write 8 digits at least to get what we want,

$$x_2 \approx 1.16666667$$

$$x_3 \approx 1.12644368$$

$$x_4 \approx 1.12249707$$

$$x_5 \approx 1.12246205$$

$$x_6 \approx 1.12246205$$

Since x_5 and x_6 agree to eight decimal places, we conclude that

$$\sqrt[6]{2} \approx 1.12246205$$

to eight decimal places.

Example Find, correct to six decimal places, the root of the equation $\cos(x) = x$.

Solution : We first rewrite the equation in standard form

$$\cos(x) - x = 0$$

Therefore we let $f(x) = \cos(x) - x$. Then $f'(x) = -\sin(x) - 1$, so formula (*) becomes

$$x_{n+1} = x_n - \frac{\cos(x_n) - x_n}{-\sin(x_n) - 1} = x_n + \frac{\cos(x_n) - x_n}{\sin(x_n) + 1}$$

in order to guess a suitable value for x_1 we sketch the graphs of $y = \cos(x)$ and $y = x$:

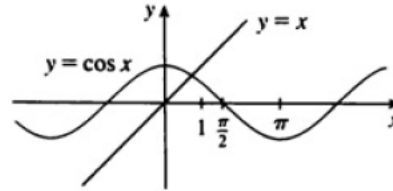


FIGURE 6

It appears that they intersect a point whose x -coordinate is somewhat less than 1. So let's take $x_1 = 1$, as a convenient first approximation. Then, remembering to put our calculator in radian mode we get

$$x_2 \approx 0.75036387$$

$$x_3 \approx 0.73911289$$

$$x_4 \approx 0.73908513$$

$$x_5 \approx 0.73908513$$

Since x_4 and x_5 agree to 6 decimal places (eight in fact), we conclude that the root of the equation, correct to six decimal places is 0.739085.

(Note that better the approximation x_1 is, faster the Newton Method is, we could have also used the a graph given by a calculator instead and the Newton Method might have converges much faster.)